

# Uniform asymptotics for the tail probability of weighted sums with heavy tails

Chenhua Zhang <sup>1</sup>

**Abstract.** This paper studies the tail probability of weighted sums of the form  $\sum_{i=1}^n c_i X_i$ , where random variables  $X_i$ 's are either independent or pairwise quasi-asymptotical independent with heavy tails. Using  $h$ -insensitive function, the uniform asymptotic equivalence of the tail probabilities of  $\sum_{i=1}^n c_i X_i$ ,  $\max_{1 \leq k \leq n} \sum_{i=1}^k c_i X_i$  and  $\sum_{i=1}^n c_i X_i^+$  is established, where  $X_i$ 's are independent and follow the long-tailed distribution, and  $c_i$ 's take value in a broad interval. Some further uniform asymptotic results for the weighted sums of  $X_i$ 's with dominated varying tails are obtained. An application to the ruin probability in a discrete-time insurance risk model is presented.

MSC: 41A60; 62P05; 62E20; 91B30

**Keywords:**  $h$ -insensitive function, long-tailed distribution, consistently varying tail, dominated variation, quasi-asymptotical independence

## 1. Introduction

In this paper, all asymptotic and limit relations are taken as  $x \rightarrow \infty$  unless otherwise stated. For independently and identically distributed (iid) subexponential random variables  $X_i, i \geq 1$ , it is well-known that, for any  $n \geq 2$ ,

$$P\left(\sum_{i=1}^n X_i > x\right) \sim P\left(\max_{1 \leq k \leq n} \sum_{i=1}^k X_i > x\right) \sim P\left(\sum_{i=1}^n X_i^+ > x\right) \sim \sum_{i=1}^n P(X_i > x), \quad (1)$$

where  $x^+ = \max\{x, 0\}$ . There are quite a few ways to generalize these asymptotic relations. One way is to consider some broader classes of heavy-tailed distributions, see, e.g., Ng et al. [18]. Another way is to study the randomly stopped sums, see, e.g., Denisov et al. [6]. Allowing some dependence of  $X_i$ 's, similar results can be obtained for different classes of heavy-tailed distributions, see Wang and Tang [22], Geluk and Ng [11], Tang [20], Geluk and Tang [12], and references therein.

A more general way is to work on the weighted sums of form  $\sum_{i=1}^n c_i X_i$ , where weights  $c_i$ 's are real numbers. If  $X_i$ 's are iid subexponential random variables, Tang and Tsitsiashvili [21] proved that for any  $0 < a \leq b < \infty$ , the asymptotic relation

$$P\left(\sum_{i=1}^n c_i X_i > x\right) \sim \sum_{i=1}^n P(c_i X_i > x), \quad (2)$$

holds uniformly for  $a \leq c_i \leq b, 1 \leq i \leq n$ , in the sense that

$$\lim_{x \rightarrow \infty} \sup_{a \leq c_i \leq b, 1 \leq i \leq n} \left| \frac{P(\sum_{i=1}^n c_i X_i > x)}{\sum_{i=1}^n P(c_i X_i > x)} - 1 \right| = 0.$$

---

<sup>1</sup>Department of Mathematics, The University of Southern Mississippi, Hattiesburg, MS 39406-5045, USA, chen-hua.zhang@usm.edu

Recently, Liu et al. [16] and Li [14] established the same asymptotic relation for some dependent  $X_i$ 's.

Chen et al. [3] showed that for any fixed  $0 < a \leq b < \infty$  it holds that uniformly for  $a \leq c_i \leq b$ ,  $1 \leq i \leq n$ ,

$$P\left(\sum_{i=1}^n c_i X_i > x\right) \sim P\left(\max_{1 \leq k \leq n} \sum_{i=1}^k c_i X_i > x\right) \sim P\left(\sum_{i=1}^n c_i X_i^+ > x\right), \quad (3)$$

where  $X_i$ 's are independent, not necessarily identically distributed, random variables with long-tailed distributions. This result is extended by substituting  $b$  with any positive function  $b(x)$  such that  $h(x) \nearrow \infty$  and  $b(x) = o(x)$  in this paper.

Replacing the constant weights  $c_i$ 's with random weights  $\theta_i$ 's, the asymptotic relation (2) and (3) still hold if the weights  $\theta_i$ 's, independent of  $X_i$ 's, are uniformly bounded away from zero and infinity. Then it is very natural to consider the randomly weighted sum of form  $\sum_{i=1}^n \theta_i X_i$ . Wang and Tang [23] obtained  $P\left(\sum_{i=1}^n \theta_i X_i > x\right) \sim P\left(\max_{1 \leq k \leq n} \sum_{i=1}^k \theta_i X_i > x\right) \sim P\left(\sum_{i=1}^n \theta_i X_i^+ > x\right)$  for the case that the random weights are not necessarily bounded and  $X_i$ 's are independently random variables with common distribution belonging to a smaller class than the class of subexponential distributions. Furthermore, Zhang et al. [24], Chen and Yuen [4] established the same results for dependent  $X_i$ 's, where the dependence structures of  $X_i$ 's are essentially same for proof of their results.

The rest of this paper is organized as follows. Section 2 reviews some important classes of heavy-tailed distributions. Section 3 states the main results along with some corollaries. Section 4 gives an application of the main results to the ruin probability in a discrete-time insurance risk model. The proof of the main results and some lemmas are presented in Section 5.

## 2. Classes of Heavy-Tailed Distributions

A random variable  $X$  or its distribution  $F$  is said to be heavy-tailed to the right or have a heavy (right) tail if the corresponding moment generate function does not exist on the positive real line, i.e.,  $Ee^{tX} = \int_{-\infty}^{\infty} e^{tx} dF(x) = \infty$  for any  $t > 0$ . The most important class of heavy-tailed distributions is the class of subexponential distributions, denoted by  $\mathcal{S}$ . Write the tail distribution by  $\overline{F}(x) = 1 - F(x)$  for any distribution  $F$ . Let  $F^{*n}$  denote the  $n$ -fold convolution of  $F$ . A distribution  $F$  concentrated on  $[0, \infty)$  is subexponential if

$$\overline{F^{*n}}(x) \sim n\overline{F}(x)$$

for some or, equivalently, for all  $n \geq 2$ . More generally, a distribution  $F$  on  $(-\infty, \infty)$  belongs to the subexponential class if  $F^+(x) = F(x)I_{\{x \geq 0\}}$  does.

Closely related to the subexponential class  $\mathcal{S}$ , the class  $\mathcal{D}$  of dominated varying distributions consists of distributions satisfying

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}(yx)}{\overline{F}(x)} < \infty$$

for some or, equivalently, for all  $0 < y < 1$ . A slightly smaller class of  $\mathcal{D}$  is the class of distributions with consistently varying tail, denoted by  $\mathcal{C}$ . Say that a distribution  $F$  belongs to the class  $\mathcal{C}$  if

$$\lim_{y \searrow 1} \liminf_{x \rightarrow \infty} \frac{\overline{F}(yx)}{\overline{F}(x)} = 1 \text{ or, equivalently, } \lim_{y \nearrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}(yx)}{\overline{F}(x)} = 1.$$

A distribution  $F$  belongs to the class  $\mathcal{L}$  of long-tailed distributions if

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1$$

for some or, equivalently, for all  $y$ . A tail distribution  $\overline{F}$  is called  $h$ -insensitive if  $\overline{F}(x+y) \sim \overline{F}(x)$  holds uniformly for all  $|y| \leq h(x)$ , where  $h(x)$  is a positive nondecreasing function and  $\lim_{x \rightarrow \infty} h(x) = \infty$ . The concept of  $h$ -insensitive function is extensively used in the monograph of Foss et al. [9]. For any distribution  $F \in \mathcal{L}$ , it can be shown that  $\overline{F}$  is  $h$ -insensitive for some positive nondecreasing function  $h(x) := h_F(x)$  such that  $h(x) \nearrow \infty$  and  $h(x) = o(x)$ , see, e.g., Lemma 5.1 in Section 5, Section 2 in Foss and Zachary [10], Lemma 4.1 of Li et al. [15]. Consequently,  $\overline{F}$  is  $ch$ -insensitive for any fixed positive real number  $c$ .

It is known that the proper inclusion relations

$$\mathcal{C} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$$

hold, see, e.g., Embrechts et al. [8], Foss et al. [9].

### 3. Main Results

Throughout the rest of this paper  $X_i, i \geq 1$ , are random variables with distribution  $F_i, i \geq 1$ , respectively. Adopt the notation  $M_c F$  and  $*_{1 \leq i \leq n} M_{c_i} F_i$  in Barbe and McCormick [1]. For  $X \sim F$  and  $c > 0$ , let  $M_c F(x) = F(x/c)$  be the distribution of  $cX$ . The distribution of  $\sum_{i=1}^n c_i X_i$  is  $*_{1 \leq i \leq n} M_{c_i} F_i$ , where  $X_i, 1 \leq i \leq n$ , are independent random variables and  $*_{1 \leq i \leq n} M_{c_i} F_i$  is the convolution of  $M_{c_i} F_i, 1 \leq i \leq n$ .

The first main result generalizes Lemma 4.1 of Chen et al. [3] with different approach in two ways. First, it increases the upper bound of the weights and decreases the lower bound of the weights. Second, the fixed shift term  $A$  in Lemma 4.1 of Chen et al. [3] is enlarged to some unbounded function, which is irrespective of the upper bound of the weights.

**Theorem 3.1.** *If  $X_i \sim F_i \in \mathcal{L}, 1 \leq i \leq n$ , are independent random variables, there exists a positive nondecreasing function  $h(x) := h(x; F_1, \dots, F_n)$  satisfying  $h(x) \nearrow \infty$  such that  $*_{1 \leq i \leq n} M_{c_i} F_i$  is uniformly  $h(x)$ -long-tailed for  $a(x) \leq c_i \leq b(x), 1 \leq i \leq n$ , in the sense that*

$$P\left(\sum_{i=1}^n c_i X_i > x \pm h(x)\right) \sim P\left(\sum_{i=1}^n c_i X_i > x\right)$$

holds uniformly for  $a(x) \leq c_i \leq b(x), 1 \leq i \leq n$ , i.e.,

$$\lim_{x \rightarrow \infty} \sup_{a(x) \leq c_i \leq b(x), 1 \leq i \leq n} \left| \frac{*_{1 \leq i \leq n} M_{c_i} F_i(x \pm h(x))}{*_{1 \leq i \leq n} M_{c_i} F_i(x)} - 1 \right| = 0, \quad (4)$$

where the positive function  $b(x)$  satisfies  $b(x) \nearrow \infty$  and  $b(x) = o(x)$ ,  $h(x)$  is irrespective of  $b(x)$ ,  $a(x) = h^{-\delta}(x) \searrow 0$  for some  $\delta > 0$ .

**Remark 3.1.** Considering the case of Weibull distribution  $F_1(x) = 1 - e^{-cx^\tau} \in \mathcal{S} \subset \mathcal{L}$  with  $0 < \tau < 1$ , it indicates that the restriction on  $a(x)$  can not be weakened in general.

It is known that the class  $\mathcal{L}$  is closed under convolution (see, e.g., Theorem 3 of Embrechts and Goldie [7], Corollary 2.42 of Foss et al. [9]), which can be also derived directly from Theorem 3.1.

**Corollary 3.1.** *If  $X_i \sim F_i \in \mathcal{L}, 1 \leq i \leq n$ , are independent random variables, then the distribution of  $\sum_{i=1}^n c_i X_i > x$  is long-tailed for any fixed  $c_i > 0, 1 \leq i \leq n$ . Consequently, the class  $\mathcal{L}$  of long-tailed distributions is closed under convolution.*

**Theorem 3.2.** *If  $X_i \sim F_i \in \mathcal{L}, 1 \leq i \leq n$ , are independent random variables, there exist positive functions  $a(x)$  and  $b(x)$  satisfying  $a(x) \searrow 0$  and  $b(x) \nearrow \infty$  such that the asymptotic relations (3) hold uniformly for  $a(x) \leq c_i \leq b(x), 1 \leq i \leq n$ .*

The following result can be also founded in Lemma 3.4 of Foss et al. [9].

**Corollary 3.2.** *A distribution  $F \in \mathcal{S}$  iff  $F \in \mathcal{L}$  and  $\overline{F} * \overline{F}(x) \sim 2\overline{F}(x)$ .*

Random variables  $X_i, i \geq 1$ , are pairwise strong quasi-asymptotically independent (pSQAI) if, for any  $i \neq j$ ,

$$\lim_{\min\{x_i, x_j\} \rightarrow \infty} P(|X_i| > x_i | X_j > x_j) = 0,$$

which was used in Geluk and Tang [12], Liu et al. [16] and Li [14], and related to what is called asymptotic independence; see e.g. Resnick [17].

**Theorem 3.3.** *If  $X_i \sim F_i \in \mathcal{C}, 1 \leq i \leq n$ , are pSQAI random variables and  $b(x)$  is an arbitrary fixed positive function satisfying  $b(x) \nearrow \infty$  and  $b(x) = o(x)$ , then it holds that, uniformly for any  $0 < c_i \leq b(x), 1 \leq i \leq n$ ,*

$$P\left(\sum_{i=1}^n c_i X_i > x\right) \sim P\left(\max_{1 \leq k \leq n} \sum_{i=1}^k c_i X_i > x\right) \sim P\left(\sum_{i=1}^n c_i X_i^+ > x\right) \sim \sum_{i=1}^n P(c_i X_i > x). \quad (5)$$

**Corollary 3.3.** *Under assumption of Theorem 3.3, the above result still holds for  $0 \leq c_i \leq b(x), 1 \leq i \leq n$ , and  $\min_{1 \leq i \leq n} c_i > 0$ .*

The next theorem extends Lemma 2.1 of Liu et al [16] and Theorem 2.1 of Li [14] with a different proof, which is based on Theorem 3.1.

**Theorem 3.4.** *If  $X_i \sim F_i \in \mathcal{D} \cap \mathcal{L}, 1 \leq i \leq n$ , are pSQAI random variables, there exist a positive function  $a(x) \searrow 0$  and a positive function  $b(x) \nearrow \infty$  such that (5) holds uniformly for  $a(x) \leq c_i \leq b(x), 1 \leq i \leq n$ .*

**Remark 3.2.** Both  $a(x)$  and  $b(x)$  depend on  $h(x)$  in Theorem 3.2 and 3.4, where  $h(x) = o(x)$  is given in Theorem 3.1. More specifically,  $a(x) = h^{-\delta}(x)$  for some  $\delta > 0$  and  $b(x) = o(h(x))$ , for example,  $b(x) = h^{1/2}(x)$ .

**Remark 3.3.** If the constant weights  $c_i, 1 \leq i \leq n$  are replaced by random weights  $\theta_i, 1 \leq i \leq n$ , which are independent of  $X_i, 1 \leq i \leq n$ , conditioning on the random weights can easily establish the corresponding results for random weights sums.

The proof of Theorem 3.4 gives an extension of Lemma 4.3 of Geluk and Tang [12].

**Corollary 3.4.** *If  $X_i \sim F_i \in \mathcal{L}$ ,  $1 \leq i \leq n$ , are pQSAI random variables, it holds that, for some the positive functions  $b(x) \nearrow \infty$  and  $a(x) \searrow 0$ ,*

$$\lim_{x \rightarrow \infty} \inf_{a(x) \leq c_i \leq b(x), 1 \leq i \leq n} \frac{P(\sum_{i=1}^n c_i X_i > x)}{\sum_{i=1}^n P(c_i X_i > x)} \geq 1. \quad (6)$$

#### 4. Application to Risk Theory

Consider the following discrete-time insurance risk model

$$U_0 = x, \quad U_n = U_{n-1}(1 + r_n) - X_n, \quad n \geq 1,$$

where  $U_n$  stands an insurer's surplus at the end of period  $n$  with a deterministic initial surplus  $x$ ,  $r_n$  represents the constant interest force of an insurer's risk-free investment, and the net loss  $X_n$  over period  $n$  equals the total amount of claims plus other costs minus the total amount of premiums during period  $n$ . It is an interesting and important problem arising from the above discrete-time insurance risk model to study the ruin probabilities of the insurer. See Tang [19] for detailed discussion.

The ruin probability by time  $n$  is defined as

$$\psi(x; n) = P\left(\min_{i=1}^n U_i < 0 \mid U_0 = x\right).$$

It is easy to see that the surplus process is of form

$$U_0 = x, \quad U_n = \prod_{i=1}^n (1 + r_i)x - \sum_{i=1}^n \left( \prod_{j=i+1}^n (1 + r_j) \right) X_i, \quad n \geq 1.$$

Define the discounted surplus process as follows

$$\tilde{U}_n = \left( \prod_{i=1}^n (1 + r_i) \right)^{-1} U_n = x - \sum_{i=1}^n c_i X_i,$$

where  $c_i = \prod_{j=1}^i (1 + r_j)^{-1}$  represents the discount factor from time  $i$  to time 0,  $1 \leq i \leq n$ . Then the corresponding ruin probability can be written as

$$\psi(x; n) = P\left(\min_{i=1}^n \tilde{U}_i < 0 \mid \tilde{U}_0 = x\right) = P\left(\max_{1 \leq i \leq k} \sum_{i=1}^k c_i X_i > x\right).$$

Applying Theorem 3.2 and Theorem 3.4 in Section 3, the following asymptotic results can be obtained.

**Corollary 4.1.** *Assume that net losses  $X_i, i \geq 1$  are independent random variables, which are not necessarily identically distributed, with distribution  $F_i, i \geq 1$ , respectively. If  $F_i \in \mathcal{L}, 1 \leq i \leq n$ , then*

$$\psi(x; n) \sim P\left(\sum_{i=1}^n c_i X_i > x\right) \sim P\left(\sum_{i=1}^n c_i X_i^+ > x\right).$$

If  $F_i \in \mathcal{D} \cap \mathcal{L}, 1 \leq i \leq n$ , then

$$\psi(x; n) \sim P\left(\sum_{i=1}^n c_i X_i > x\right) \sim P\left(\sum_{i=1}^n c_i X_i^+ > x\right) \sim \sum_{i=1}^n P(c_i X_i > x).$$

## 5. Proof of Results

A function  $h(x)$  is called slowly varying at infinity if  $h(xy) \sim h(x)$  for any  $y > 0$ . It is well-known that  $h(x) = o(x^\delta)$  for any  $\delta > 0$  if  $h(x)$  is a slowly varying function, see, e.g., Bingham et al. [2]. The following result is crucial for the proof of all theorems in this paper. It shows that any tail distribution of a long-tailed distribution is uniformly  $h$ -insensitive for a slowly varying function  $h$ .

**Lemma 5.1.** *If  $X \sim F \in \mathcal{L}$ , then  $\bar{F}$  is  $h$ -insensitive for a positive nondecreasing and slowly varying function  $h(x) := h(x; F) : (0, \infty) \rightarrow (0, \infty)$  satisfying  $h(x) \nearrow \infty$ ,  $h(x) \leq ch(\frac{x}{c})$  for all  $c \geq 1$ , and*

$$\lim_{x \rightarrow \infty} \sup_{a(x) \leq c \leq b(x)} \left| \frac{P(cX > x \pm h(x))}{P(cX > x)} - 1 \right| = 0, \quad (7)$$

where  $b(x)$  is an arbitrary positive function such that  $b(x) \nearrow \infty$  and  $b(x) = o(x)$ , and  $a(x) = h^{-\delta}(x)$  for some  $\delta > 0$ .

**Proof.** For any fixed  $\delta > 0$ , let  $\{x_n, n \geq 1\}$  be a sequence of increasing positive real numbers such that  $x_{n+1} \geq 2x_n > 0$ ,  $n \geq 1$ , and for any  $x \geq x_n$ ,

$$\sup_{|y| \leq n} \left| \frac{\bar{F}(x+y)}{\bar{F}(x)} - 1 \right| \leq \max \left\{ \left| \frac{\bar{F}(x+n^{1+\delta})}{\bar{F}(x)} - 1 \right|, \left| \frac{\bar{F}(x-n^{1+\delta})}{\bar{F}(x)} - 1 \right| \right\} \leq \frac{1}{n}. \quad (8)$$

Borrowing the idea of the proof of Corollary 2.5 in [5], let

$$h(x) = \begin{cases} \frac{2}{x_1}x & x_0 = 0 < x < x_1 \\ n + \frac{x-x_{n-1}}{x_n-x_{n-1}} & x_{n-1} \leq x < x_n, n \geq 2. \end{cases}$$

Clearly,  $h(x)$  is a positive nondecreasing, piecewise linear, continuous function and  $h(x) \nearrow \infty$ . Since  $h(x)$  is a nondecreasing function,  $h(xy) \sim h(x)$  for any  $y > 0$  is equivalent to  $h(2x) \sim h(x)$ , which follows from the facts that  $h(x) \nearrow \infty$  and  $h(x) \leq h(2x) < h(x_{n+1}) = n+2 \leq h(x) + 2$  for any  $x_{n-1} \leq x < x_n$ .

For any  $x \geq x_n$ , i.e.,  $x \in [x_{n+k}, x_{n+k+1})$  for some  $k := k(x) \geq 0$ , and  $|y| \leq h^{1+\delta}(x) = (n+k+1)^{1+\delta}$ , it follows from (8) that

$$\sup_{|y| \leq h^{1+\delta}(x)} \left| \frac{\bar{F}(x+y)}{\bar{F}(x)} - 1 \right| \leq \frac{1}{n+k+1} \leq \frac{1}{n} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

i.e.,  $\bar{F}$  is  $h^{1+\delta}$ -insensitive, which of course implies that  $\bar{F}$  is  $h$ -insensitive. Since  $x_{n+1} - x_n \geq x_n \geq x_n - x_{n-1}$ ,  $n \geq 1$ ,  $h'(x)$  is a nonincreasing function on  $\cup_{n=1}^{\infty} (x_{n-1}, x_n)$ , which implies that  $h(x)$  is a concave function on  $[0, \infty)$ . The concavity of  $h(x)$  and the fact  $h(0) = 0$  lead to  $h(\frac{x}{c}) = h(\frac{1}{c}x + (1 - \frac{1}{c})0) \geq \frac{1}{c}h(x) + (1 - \frac{1}{c})h(0) = \frac{1}{c}h(x)$ , i.e.,  $h(x) \leq ch(\frac{x}{c})$ , for any  $x > 0, c > 1$ . Hence,  $\frac{h(x)}{c} \leq h(\frac{x}{c}) \leq h^{1+\delta}(\frac{x}{c})$  for  $1 \leq c \leq b(x)$ . Note that  $\frac{h(x)}{c} \leq \frac{h(x)}{a(x)} = h^{1+\delta}(x) \leq h^{1+\delta}(\frac{x}{c})$  for  $a(x) \leq c \leq 1$ . The monotonicity of  $\bar{F}$  yields  $\bar{F}(\frac{x}{c} + h^{1+\delta}(\frac{x}{c})) \leq P(cX > x \pm h(x)) = \bar{F}(\frac{x}{c} \pm \frac{h(x)}{c}) \leq \bar{F}(\frac{x}{c} - h^{1+\delta}(\frac{x}{c}))$  for  $a(x) \leq c \leq b(x)$ . The uniform asymptotic relation (7) follows from the inequalities

$$\begin{aligned} \frac{\bar{F}(\frac{x}{c} + h^{1+\delta}(\frac{x}{c}))}{\bar{F}(\frac{x}{c})} - 1 &\leq \frac{P(cX > x \pm h(x))}{P(cX > x)} - 1 = \frac{\bar{F}(\frac{x}{c} \pm \frac{h(x)}{c})}{\bar{F}(\frac{x}{c})} - 1 \\ &\leq \frac{\bar{F}(\frac{x}{c} - h^{1+\delta}(\frac{x}{c}))}{\bar{F}(\frac{x}{c})} - 1, \quad a(x) \leq c \leq b(x), \end{aligned}$$

and the fact that  $\overline{F}$  is  $h^{1+\delta}$ -insensitive.  $\square$

**Remark 5.1.** It is easy to show that  $\frac{h(x)}{x} \searrow 0$  for  $h(x)$  in the proof of Lemma 5.1.

**Proof of Theorem 3.1.** Assume that  $\overline{F}_i$  is  $h_i$ -insensitive, where  $h_i(x) = h(x; F_i)$  is given in Lemma 5.1,  $1 \leq i \leq n$ . Let  $h(x) := h(x; F_1, \dots, F_n) = \min\{h_i(x), 1 \leq i \leq n\} = o(x)$ . Then all  $\overline{F}_i$ 's are  $h$ -insensitive and  $h(x) \leq ch(\frac{x}{c})$ ,  $c \geq 1$ , by Lemma 5.1. The uniform asymptotic relation (6), which is essentially the case of  $n = 2$  in proof, will be proved by induction. It is obviously true for  $n = 1$  by Lemma 5.1. Since distribution functions are nondecreasing, (6) is equivalent to

$$\lim_{x \rightarrow \infty} \inf_{a(x) \leq c_i \leq b(x), 1 \leq i \leq n} \frac{P\left(\sum_{i=1}^n c_i X_i > x + h(x)\right)}{P\left(\sum_{i=1}^n c_i X_i > x\right)} \geq 1, \quad (9)$$

and

$$\lim_{x \rightarrow \infty} \sup_{a(x) \leq c_i \leq b(x), 1 \leq i \leq n} \frac{P\left(\sum_{i=1}^n c_i X_i > x - h(x)\right)}{P\left(\sum_{i=1}^n c_i X_i > x\right)} \leq 1. \quad (10)$$

Write  $A + B + C$  for the union of disjoint sets  $A, B, C$ . The fact that  $\left\{\sum_{i=1}^n c_i X_i > x \pm h(x)\right\} = \left\{\sum_{i=1}^n c_i X_i > x + h(x), c_n X_n \leq \frac{x+h(x)}{2}\right\} + \left\{\sum_{i=1}^n c_i X_i > x + h(x), \sum_{i=1}^{n-1} c_i X_i \leq \frac{x+h(x)}{2}\right\} + \left\{\sum_{i=1}^{n-1} c_i X_i > \frac{x+h(x)}{2}, c_n X_n > \frac{x+h(x)}{2}\right\}$  and independence of  $X_i$ 's yield

$$\begin{aligned} P\left(\sum_{i=1}^n c_i X_i > x + h(x)\right) &\geq \int_{-\infty}^{x/2} P\left(\sum_{i=1}^{n-1} c_i X_i > x + h(x) - t\right) dP(c_n X_n \leq t) \\ &\quad + \int_{-\infty}^{x/2} P(c_n X_n > x + h(x) - t) dP\left(\sum_{i=1}^{n-1} c_i X_i \leq t\right) \\ &\quad + P\left(\sum_{i=1}^{n-1} c_i X_i > \frac{x+h(x)}{2}\right) P\left(c_n X_n > \frac{x+h(x)}{2}\right). \end{aligned} \quad (11)$$

The induction assumption with  $b(x)$  replaced by  $2b(x)$  implies that

$$\begin{aligned} &P\left(\sum_{i=1}^{n-1} c_i X_i > \frac{x+h(x)}{2}\right) P\left(c_n X_n > \frac{x+h(x)}{2}\right) \\ &= P\left(\sum_{i=1}^{n-1} 2c_i X_i > x + h(x)\right) P\left(2c_n X_n > x + h(x)\right) \\ &\sim P\left(\sum_{i=1}^{n-1} 2c_i X_i > x\right) P\left(2c_n X_n > x\right) = P\left(\sum_{i=1}^{n-1} c_i X_i > \frac{x}{2}\right) P\left(c_n X_n > \frac{x}{2}\right) \end{aligned} \quad (12)$$

holds uniformly for  $a(x) \leq c_i \leq b(x)$ ,  $1 \leq i \leq n$ .

Use monotonicity of any distribution function and the inequality  $h(x) \leq 2h(\frac{x}{2})$  to obtain

$$1 \geq \inf_{t \leq x/2} \frac{\overline{F}(x + h(x) - t)}{\overline{F}(x - t)} \geq \inf_{t \leq x/2} \frac{\overline{F}(x - t + 2h(\frac{x}{2}))}{\overline{F}(x - t)} \geq \inf_{u=x-t \geq x/2} \frac{\overline{F}(u + 2h(u))}{\overline{F}(u)} \sim 1 \quad (13)$$

provided  $\overline{F}$  is  $h$ -insensitive. It follows from the induction assumption and Lemma 5.1 that the tail distribution of  $\sum_{i=1}^{n-1} c_i X_i$  and the tail distribution of  $c_n X_n$  are  $h$ -insensitive. The asymptotic

relation (12) and the inequality (11) imply

$$\begin{aligned}
& P\left(\sum_{i=1}^n c_i X_i > x + h(x)\right) \\
& \geq \left( \int_{-\infty}^{x/2} P\left(\sum_{i=1}^{n-1} c_i X_i > x - t\right) dP(c_n X_n \leq t) + \int_{-\infty}^{x/2} P(c_n X_n > x - t) dP\left(\sum_{i=1}^{n-1} c_i X_i \leq t\right) \right. \\
& \quad \left. + P\left(\sum_{i=1}^{n-1} c_i X_i > \frac{x}{2}\right) P\left(c_n X_n > \frac{x}{2}\right) \right) (1 + o(1)) \\
& = (1 + o(1)) P\left(\sum_{i=1}^n c_i X_i > x\right),
\end{aligned}$$

where the term  $o(1)$  goes to 0 uniformly for  $a(x) \leq c_i \leq b(x)$ ,  $1 \leq i \leq n$ . This complete the proof of (9).

The other uniform asymptotic relation (10) can be obtained by substituting  $+h(x)$ ,  $+2h(\frac{x}{2})$ ,  $\geq$ ,  $\inf$  with  $-h(x)$ ,  $-2h(\frac{x}{2})$ ,  $\leq$ ,  $\sup$ , respectively, in the proof of (9).  $\square$

**Proof of Theorem 3.2.** The idea is from the proof of Theorem 2.1 of Chen et al. [3]. Let  $\{\Omega_K = \{X_i \geq 0 \text{ for all } i \in K, X_j < 0 \text{ for all } j \in \{1, \dots, n\} \setminus K\}, K \subseteq \{1, \dots, n\}\}$  be a finite partition of the whole space  $\Omega$ . Obviously,  $P(\sum_{i=1}^n c_i X_i > x, \Omega_K)$  is not less than

$$\begin{aligned}
& P\left(\sum_{i \in K} c_i X_i > x + h(x), \sum_{j \notin K} c_j X_j > -h(x), \Omega_K\right) \\
& = P\left(\sum_{i=1}^n c_i X_i^+ > x + h(x), \Omega_K\right) - P\left(\sum_{i \in K} c_i X_i > x + h(x), \sum_{j \notin K} c_j X_j \leq -h(x), \Omega_K\right), \quad (14)
\end{aligned}$$

where, due to the independence of  $X_i$ 's, the second term equals

$$P\left(\sum_{i \in K} c_i X_i > x + h(x), \bigcap_{i \in K} \{X_i \geq 0\}\right) P\left(\sum_{j \notin K} c_j (-X_j) \geq h(x), \bigcap_{j \notin K} \{X_j < 0\}\right).$$

and it is at most  $P(\sum_{i=1}^n c_i X_i^+ > x + h(x)) P(\sum_{j=1}^n c_j X_j^- \geq h(x))$ , where  $x^- = \max\{-x, 0\}$ . Note that  $\{\sum_{j=1}^n c_j X_j^- \geq h(x)\} \subseteq \bigcup_{j=1}^n \{c_j X_j^- \geq \frac{h(x)}{n}\} = \bigcup_{j=1}^n \{c_j X_j \leq -\frac{h(x)}{n}\}$ , whose probability is at most  $\sum_{j=1}^n P(X_j \leq -\frac{h(x)}{nb(x)}) = o(1)$  provided  $b(x) = o(h(x))$ . Therefore, uniformly for  $0 < a \leq c_i \leq b(x)$ ,  $1 \leq i \leq n$ , the second term in (14) is  $o(P(\sum_{i=1}^n c_i X_i^+ > x + h(x)))$  and

$$P\left(\sum_{i=1}^n c_i X_i > x, \Omega_K\right) \geq P\left(\sum_{i=1}^n c_i X_i^+ > x + h(x), \Omega_K\right) + o\left(P\left(\sum_{i=1}^n c_i X_i^+ > x + h(x)\right)\right).$$

Sum it over all  $K$ 's to get

$$P\left(\sum_{i=1}^n c_i X_i > x\right) \geq P\left(\sum_{i=1}^n c_i X_i^+ > x + h(x)\right) + o\left(P\left(\sum_{i=1}^n c_i X_i^+ > x + h(x)\right)\right).$$

Clearly,  $X_i^+ \sim F_i^+(x) = F_i(x)I_{\{x \geq 0\}} \in \mathcal{L}$ ,  $1 \leq i \leq n$ . Choose  $h(x)$  such that (6) holds with  $F_i$  substituted by  $F_i^+$ . The desired result follows from Theorem 3.1 and the simple fact that  $\sum_{i=1}^n c_i X_i \leq \max_{1 \leq k \leq n} \sum_{i=1}^k c_i X_i \leq \sum_{i=1}^n c_i X_i^+$ .  $\square$



**Proof of Corollary 3.2.** Recall that  $\overline{F} \in \mathcal{S}$  if  $\overline{F^+} \in \mathcal{S}$ , i.e.,  $\overline{F^+ * F^+}(x) \sim 2\overline{F^+}(x)$  for  $F^+(x) = F(x)I_{\{x \geq 0\}}$ . Clearly,  $F \in \mathcal{L}$  iff  $F^+ \in \mathcal{L}$ . If  $F^+ \in \mathcal{S}$ , the fact that  $\mathcal{S} \subset \mathcal{L}$  implies  $F \in \mathcal{L}$ . Then it is equivalent to show that  $\overline{F^+ * F^+}(x) \sim 2\overline{F^+}(x)$  iff  $\overline{F * F}(x) \sim 2\overline{F}(x)$ , i.e.  $\overline{F^+ * F^+}(x) \sim \overline{F * F}(x)$  since  $\overline{F^+}(x) = \overline{F}(x)$  for all  $x > 0$ . It is obviously true by Theorem 3.2.  $\square$

The next two lemma can be easily checked from the definition of the class  $\mathcal{C}$ .

**Lemma 5.2.** *If  $X$  follows distribution  $F \in \mathcal{C}$ , then  $\overline{F}(x)$  is  $h$ -insensitive provided  $h(x) = o(x)$  and it holds that, uniformly for  $0 < c < b(x) = o(x)$ ,*

$$P(cX > x \pm h(x)) \sim P(cX > x).$$

**Lemma 5.3.** *If  $X_i \sim F_i \in \mathcal{C}$ ,  $1 \leq i \leq n$ , are pQSAI random variables, it holds that, uniformly for  $0 < c < b(x) = o(x)$ ,*

$$P\left(c_j X_j > \frac{x}{n}, \max_{1 \leq k \neq j \leq n} |c_k X_k| > b(x) \ln\left(\frac{x}{b(x)}\right)\right) = o(P(c_j X_j > x))$$

and consequently

$$P\left(\bigcup_{j=1}^n \left\{c_j X_j > \frac{x}{n}, \max_{1 \leq k \neq j \leq n} |c_k X_k| > b(x) \ln\left(\frac{x}{b(x)}\right)\right\}\right) = o\left(\sum_{j=1}^n P(c_j X_j > x)\right).$$

**Proof of Theorem 3.3.** Let  $h(x) = b(x) \ln\left(\frac{x}{b(x)}\right)$ . The proof is similar to that of Theorem 3.4 and is omitted.  $\square$

**Proof of Corollary 3.3.** Partition the range of the weights as  $\{(c_1, \dots, c_n) : 0 \leq c_i \leq b(x), 1 \leq i \leq n, \min_{i=1}^n c_i > 0\} = \bigcup_{K \subset \{1, \dots, n\}} \{(c_1, \dots, c_n) : 0 \leq c_i \leq b(x), i \in K, 0 < c_i \leq b(x), i \notin K\}$ . The desired result follows from Theorem 3.3.  $\square$

**Lemma 5.4.** *If  $X_i \sim F_i \in \mathcal{D}$ ,  $1 \leq i \leq n$ , are pSQAI random variables,  $h(x) = o(x)$  and  $h(x) \nearrow \infty$ , it holds that, uniformly for  $0 < a < c_i < b(x) = o(h(x))$ ,  $1 \leq i \leq n$ ,*

$$P\left(c_j X_j > \frac{x}{n}, \max_{1 \leq k \neq j \leq n} |c_k X_k| > h(x)\right) = o(P(c_j X_j > x))$$

and consequently

$$P\left(\bigcup_{j=1}^n \left\{c_j X_j > \frac{x}{n}, \max_{1 \leq k \neq j \leq n} |c_k X_k| > h(x)\right\}\right) = o\left(\sum_{j=1}^n P(c_j X_j > x)\right).$$

**Proof.** The results follow from the fact that  $F_i \in \mathcal{D}$  and  $b(x) = o(h(x))$ , the pSQAI property of  $X_i$ 's and the elementary probability inequality  $P(A \cap \bigcup_{i=1}^n B_i) \leq \sum_{i=1}^n P(AB_i)$ .  $\square$

If  $X_i$  is large, the pSQAI property of  $X_j$ 's implies that other  $X_j$ 's are relatively close to 0 and negligible compared with  $X_i$ . If  $\sum_{i=1}^n c_i X_i > x$ , there should be exactly one  $c_i X_i$  greater than  $\frac{x}{n}$  and consequently Lemma 5.4 implies

$$P\left(\sum_{i=1}^n c_i X_i > x\right) \sim \sum_{j=1}^n P\left(\sum_{i=1}^n c_i X_i > x, c_j X_j > \frac{x}{n}, \max_{1 \leq k \neq j \leq n} |c_k X_k| \leq h(x)\right).$$

It gives the idea of the proof of Theorem 3.4, which is simpler and more straightforward than the proof of Lemma 2.1 of Liu et al. [16] and Theorem 2.1 of Li [14].

**Proof of Theorem 3.4.** All asymptotic relations hold uniformly for  $a(x) \leq c_i \leq b(x)$ ,  $1 \leq i \leq n$ , in the proof. By Lemma 5.1, there exists a positive nondecreasing function  $h(x) := h(x, a; F_1, \dots, F_n)$  satisfying  $h(x) \nearrow \infty$  and  $h(x) = o(x)$  such that (7) holds for  $F = F_i$ ,  $1 \leq i \leq n$ , respectively. Choose  $b(x) = o(h(x))$  and  $b(x) \nearrow \infty$ . Note that

$$\begin{aligned} \left\{ \sum_{i=1}^n c_i X_i > x \right\} &= \bigcup_{j=1}^n \left\{ \sum_{i=1}^n c_i X_i > x, c_j X_j > \frac{x}{n} \right\} \\ &= \bigcup_{j=1}^n A_j \bigcup \left\{ \sum_{i=1}^n c_i X_i > x, \bigcup_{j=1}^n \left\{ c_j X_j > \frac{x}{n}, \max_{1 \leq k \neq j \leq n} |c_k X_k| > h(x) \right\} \right\}, \end{aligned}$$

where  $A_j = \left\{ \sum_{i=1}^n c_i X_i > x, c_j X_j > \frac{x}{n}, \max_{1 \leq k \neq j \leq n} |c_k X_k| \leq h(x) \right\}$ ,  $1 \leq j \leq n$ , are mutually exclusive events provided  $\frac{x}{n} > h(x)$ . The elementary probability inequality  $P(A) \leq P(A \cup B) \leq P(A) + P(B)$  and Lemma 5.4 lead to

$$P\left(\sum_{i=1}^n c_i X_i > x\right) = \sum_{j=1}^n P(A_j) + o\left(\sum_{j=1}^n P(c_j X_j > x)\right). \quad (15)$$

Lemma 5.1 and the fact that  $c_j X_j$  is at least  $x - (n-1)h(x)$  on  $A_j$  lead to

$$P(A_j) \leq P(c_j X_j > x - (n-1)h(x)) = P(c_j X_j > x) + o(P(c_j X_j > x)), \quad 1 \leq j \leq n.$$

Since  $\max_{1 \leq k \neq j \leq n} |c_k X_k| \leq h(x)$  on  $A_j$ ,  $c_j X_j > x + (n-1)h(x)$  implies  $\sum_{i=1}^n c_i X_i > x$  on  $A_j$  for any  $1 \leq j \leq n$ . It follows from Lemma 5.1 and 5.4 that

$$\begin{aligned} P(A_j) &\geq P(c_j X_j > x + (n-1)h(x), \max_{1 \leq k \neq j \leq n} |c_k X_k| \leq h(x)) \\ &= P(c_j X_j > x + (n-1)h(x)) - P(c_j X_j > x + (n-1)h(x), \max_{1 \leq k \neq j \leq n} |c_k X_k| > h(x)) \\ &= P(c_j X_j > x) + o(P(c_j X_j > x)), \quad 1 \leq j \leq n. \end{aligned}$$

Therefore, (15) can be written as

$$P\left(\sum_{i=1}^n c_i X_i > x\right) \sim \sum_{i=1}^n P(c_i X_i > x). \quad (16)$$

In the exactly same way, it can be proved that

$$P\left(\sum_{i=1}^n c_i X_i^+ > x\right) \sim \sum_{i=1}^n P(c_i X_i^+ > x) = \sum_{i=1}^n P(c_i X_i > x). \quad (17)$$

Note that  $\sum_{i=1}^n c_i X_i \leq \max_{1 \leq k \leq n} \sum_{i=1}^k c_i X_i \leq \sum_{i=1}^n c_i X_i^+$ . The desired results follow from the uniform asymptotic relation (16) and (17).  $\square$

**Remark 5.2.** The proof of Theorem 3.4 also leads to Corollary 3.4.

## Acknowledgments

The author would like to thank the anonymous referees for their comments and help in improving the paper.

## References

- [1] Barbe, P. and McCormick, W.P. (2009). Asymptotic expansions for infinite weighted convolutions of heavy tail distributions and applications. *Memoirs of the American Mathematical Society*. 197, 1–117.
- [2] Bingham, N.H., Goldie, C.M., Teugels, J.L. (1989) *Regular Variation*. Cambridge University Press, Cambridge.
- [3] Chen, Y., Ng, K.W., Yuen, K.C. (2011) The maximum of randomly weighted sums with long tails in insurance and finance. *Stoch. Anal. Appl.* 29, 1033–1044.
- [4] Chen, Y., Yuen, K.C. (2009) Sums of pairwise quasi-asymptotically independent random variables with consistent variation. *Stoch. Models* 25, 76–89.
- [5] Cline, D.B.H., Samorodnitsky, G. (1994) Subexponentiality of the product of independent random variables. *Stoch. Process. Their. Appl.* 49, 75–98.
- [6] Denisov, D., Foss, S., Korshunov, D. (2010) Asymptotics of randomly stopped sums in the presence of heavy tails. *Bernoulli* 16, 971–994.
- [7] Embrechts, P., Goldie, C. (1980) On closure and factorization properties of subexponential and related distributions. *J. Austral. Math. Soc.* 29, 243–256.
- [8] Embrechts, P., Klüppelberg, C., Mikosch, T. (1997) *Modelling extremal events for insurance and finance*. Springer. Berlin.
- [9] Foss, S., Korshunov, D., Zachary, S. (2011) *An introduction to heavy-tailed and subexponential distributions*. Springer. New York.
- [10] Foss, S., Zachary, S. (2003) The maximum on a random time interval of a random walk with long-tailed increments and negative drift. *Ann. Appl. Probab.* 13, 37–53.
- [11] Geluk, J., Ng, K.W. (2006) Tail behavior of negatively associated heavy-tailed sums. *J. Appl. Probab.* 43(2), 587–593.
- [12] Geluk, J., Tang, Q. (2009) Asymptotic tail probabilities of sums of dependent subexponential random variables. *J. Theoret. Prob.* 22, 871–882.
- [13] Klüppelberg, C. (1988) Subexponential distributions and integrated tails. *J. Appl. Probab.* 25, 132–141.
- [14] Li, J. (2013) On pairwise quasi-asymptotically independent random variables and their applications. *Statist. Probab. Lett.* 83, 2081–2087.
- [15] Li, J., Tang, Q., Wu, R. (2010). Subexponential tails of discounted aggregate claims in a time-dependent renewal risk model. *Adv. Appl. Probab.* 42, 1126–1146.
- [16] Liu, X., Gao, Q., Wang, Y. (2012) A note on a dependent risk model with constant interest rate. *Statist. Probab. Lett.* 82, 707–712.

- [17] Resnick, S.I. (2002) Hidden regular variation, second order regular variation and asymptotic independence. *Extremes* 5, 303–336.
- [18] Ng, K.W., Tang, Q., Yang, H. (2002), Maxima of sums of heavy-tailed random variables. *Astin Bull.* 32, 43–55.
- [19] Tang, Q. (2004) The ruin probability of a discrete time risk model under constant interest rate with heavy tails. *Scand. Actuarial J.* 3, 229–240.
- [20] Tang, Q. (2008) Insensitivity to negative dependence of asymptotic tail probabilities of sums and maxima of sums. *Stoch. Anal. Appl.* 26, 435–450.
- [21] Tang, Q., Tsitsiashvili, G. (2003) Randomly weighted sums of subexponential random variables with application to ruin theory. *Extremes* 6, 171–188.
- [22] Wang D., Tang Q. (2004) Maxima of sums and random sums for negatively associated random variables with heavy tails. *Statist. Probab. Lett.* 68, 287–295.
- [23] Wang, D., Tang, Q. (2006) Tail probabilities of randomly weighted sums of random variables with dominated variation. *Stoch. Models* 22, 253–272.
- [24] Zhang, Y., Shen, X., Weng, C. (2009) Approximation of the tail probability of randomly weighted sums and applications. *Stoch. Proc. Appl.* 119, 655–675.